Math 255A Lecture 23 Notes

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1 The Spectral Theorem for Compact, Self-Adjoint Operators

1.1 Orthogonal projections and the resolvent

Let H be a Hilbert space, and let $T : H \to H$ be compact and self-adjoint. Then $\mathrm{Spec}(T) \subseteq \mathbb{R}$. If $\lambda \in \mathrm{Spec}(T) \setminus \{0\}$, then $R(z) = (T - zI)^{-1} = \frac{\Pi_\lambda}{\lambda - z} + \mathrm{Hol}(z)$ for $0 < |z - \lambda| < \varepsilon \ll 1$, where Π_λ is an orthogonal projection with $\mathrm{im}(\Pi_\lambda) = \ker(T - \lambda I)$.

Let λ_1, λ_2 be distinct nonzero eigenvalues of T, and notice that

$$\Pi_{\lambda_j} \Pi_{\lambda_k} = 0$$

if $j \neq k$. This follows from the fact that $\ker(-\lambda_j I) \perp \ker(T - \lambda_k I)$. It follows that the series $\sum_{j \geq 1} \pi_{\lambda_j} x$ converges in H for all x. Indeed,

$$\sum_{j=1}^N \|\Pi_{\lambda_j} x\|^2 \leq \|x\|^2$$

for each N by Bessel's inequality, so the same bound holds for the infinite sequence. Then

$$\left\| \sum_{j=1}^N \Pi_{\lambda_j} x - \sum_{j=1}^M \Pi_{\lambda_j} x \right\|^2 = \left\| \sum_{j=M+1}^N \Pi_{\lambda_j} x \right\|^2 = \sum_{j=M+1}^N \|\Pi_{\lambda_j} x\|^2 \xrightarrow{N,M \to \infty} 0$$

If we let $\Pi x = \sum_{j \geq 1} \Pi_{\lambda_j} x$, then $\Pi \in \mathcal{L}(H, H)$ is an orthogonal projection.

Proposition 1.1. *For all $x \in H$ and $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$R(z)x = (T - zI)^{-1} x = \sum_{j=1}^\infty \frac{\Pi_{\lambda_j} x}{\lambda_j - z}.$$

The series in the right hand side converges with $\| \sum_{j=1}^\infty \frac{\Pi_{\lambda_j} x}{\lambda_j - z} \| \leq \|x\| / |\mathrm{Im}(z)|$.

Proof. Consider

$$f(z) = \langle R(z)x, y\rangle - \sum_{j=0}^{\infty} \frac{\langle \Pi_{\lambda_j} x, y\rangle}{\lambda_j - z}$$

for all $x, y \in H$ and $z \in \mathbb{C}\setminus\mathbb{R}$. Then f is holomorphic on $\mathbb{C}\setminus\{0\}$, $\|f(z)| \leq 2\|x\|\|y\|/|\operatorname{Im}(z)$, and $|f(z)| \leq O(1/|z|^2)$ as $|z| \to \infty$. Indeed,

$$R(z) = (T - zI)^{-1} = ((-z)(I - T/z))^{-1} = -\frac{1}{z} I + O(1/|z|^2)$$

$$\sum_{j=1}^{\infty} \frac{\Pi_{\lambda_j}}{\lambda_j - z} = -\frac{1}{z} \underbrace{\sum \Pi_{\lambda_j}}_{=I} + I(1/|z|^2),$$

and we get the decay of f. We can write the Laurent expaansion at $z = 0$,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \qquad \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{j+1}}\, dz$$

for $0 < |z| < \infty$.

We claim that $a_j = 0$ for all j. If $j + 1 \geq 0$, let $R \gg 1$. Then $|a_j| \leq O(1/R^2)R \to 0$. So $f(z) = \sum_{j=-\infty}^{-2} a_j z^j$. If $j + 1 < 0$, then let $k = -j - 1 > 0$. Then, assuming that $\int f(z) z^{k-2}\, dz = 0$.

$$\int_{|z|=R} f(z) z^k\, dz = \int_{|z|=R} f(z) z^{k-2}(z^2 - R^2)\, dz,$$

so

$$\begin{aligned}
\left\| \int_{|z|=R} f(z) z^k\, dz \right\| &\leq 2\|x\|\|y\| \int_{|z|=R} \frac{R^{k-2}}{|\operatorname{Im}(z)}|z^2 - R^2|\, |dz| \\
&\overset{z=Rw}{=} R\frac{2\|x\|\|y\|}{R} R^{k-2} R^2 \int_{|w|=1} \frac{|w^2-1|}{|\operatorname{Im}(w)|}\, |dw| \\
&= 2\|x\|\|y\| R^k \underbrace{\int_{|w|=1} \frac{|w^2-1|}{|\operatorname{Im}(w)|}\, |dw|}_{<\infty}.
\end{aligned}$$

Letting $R \to 0$, we get $a_j = 0$ for all j. □

Remark 1.1. Observe that

$$\|R(z)x\|^2 = \sum_{j=0}^{\infty} \frac{1}{|\lambda - z|^2} \|\Pi_{\lambda_j} x\|^2.$$

We also get that

$$\|(T - zI)^{-1}\|_{\mathcal{L}(H,H)} = \frac{1}{\operatorname{dist}(z, \operatorname{Spec}(T))}.$$

This estimate remains valid for all self-adjoint $T \in \mathcal{L}(H, H)$, but we will not prove that in this course.

1.2 The missing projection

Write $\lambda_0 = 0$ and $\Pi_{\lambda_0} = I - \Pi$. Then Π_{λ_0} is an orthogonal projection.

Proposition 1.2. Π_{λ_0} *is the orthogonal projection onto* $\ker(T)$*.*

Proof. Write

$$x = (T - zI)R(z)x = \sum_{j=1}^{\infty} \frac{(T - zI)\Pi_{\lambda_j} x}{\lambda_j - z} + \frac{(T - z)\Pi_{\lambda_0} x}{-z} = \underbrace{\sum_{j=1}^{\infty} \Pi_{\lambda_j} x + \Pi_{\lambda_0} x}_{=x} - \frac{T\Pi_{\lambda_0} x}{z}$$

So $T\Pi_{\lambda_0} = 0$, and $\operatorname{im}(\Pi_{\lambda_0} \subseteq \ker(T)$. IF $x \in \ker(T)$, then

$$-x/z = R(z)x = \underbrace{\sum_{j=1}^{\infty} \frac{\Pi_{\lambda_j} x}{\lambda_j - z}}_{=0} - \frac{\Pi_{\lambda_0} x}{z}.$$

So $x = \Pi_{\lambda_0} x$, making $x \in \operatorname{im}(\Pi_{\lambda_0})$. □

We can write $x = \sum_{j=0}^{\infty} \Pi_{\lambda_j} x$ for all x, which is equivalent to $H = \bigoplus_{j \geq 0} H_j$, where $H_j = \Pi_{\lambda_j} H = \ker(T - \lambda_j I)$.

Theorem 1.1 (spectral theorem for compact, self-adjoint operators)**.** *Let* $T \in \mathcal{L}(H, H)$ *be compact and self-adjoint. Then* H *has an orthonormal basis consisting of eigenvectors of* T*.*

Proof. Choose an orthonormal basis in $H_j, j \geq 0$. □